Consistency

If we increase the sample size then our estimator is closer and closer to the parameter it is called consistency. If estimate become closer to parameter then the variance of this estimate is minimize. As $n \to \infty$ then variance of the parameter is zero.

Or

An estimator is said to be consistent only if and only if with the increase in sample size or it becomes closer and closer to parameter which is known as consistent estimator.

Or

An estimator 'θ', is said to be consistent if it tends to parameter 'θ'.
As $n \to \infty$

$$\lim_{n\to\infty} P[|\hat{\theta} - \theta| \geq \in] = 0$$

$$\lim_{n\to\infty} P[|\hat{\theta} - \theta| \leq \in] = 1 \qquad \text{Condition-I}$$

$$\lim_{n\to\infty} Var(\hat{\theta}) = 0 \qquad \text{Condition-II}$$

PROCEDURE

First we consider $\lim_{n\to\infty} P[|\hat{\theta} - \theta| \leq \in] = 1$ and convert it as $P[|z| \leq \infty] = 1$. If it is true then consider $\lim_{n\to\infty} Var(\hat{\theta})$. Then it must be equal to zero.

$$\therefore \lim_{n\to\infty} Var(\hat{\theta}) = 0$$

If these conditions are satisfied then estimator are said to be consistent.

Q.1

Show that sample mean is consistent estimator of the parameter θ.

If $x \sim N(\theta,1)$

Solution

As $X \sim N(\theta,1)$

$$\bar{X} \sim N(\theta, 1/n)$$

i.e

$$E(\bar{X}) = E(\hat{\theta}) = \theta$$

$$Var(\bar{X}) = Var(\hat{\theta}) = 1/n$$

Now we consider

$$\lim_{n\to\infty} P[|\hat{\theta} - \theta| \leq \in]$$

Replacing these values:

$$== \lim_{n\to\infty} P[|\hat{\theta} - \theta| \le \in]$$

Dividing by $\sqrt[1]{n}$ $\delta \bar{x} = \dfrac{1}{\sqrt{n}}$

$$= \lim_{n\to\infty} p[\frac{|\hat{\theta}-\theta|}{\frac{1}{\sqrt{n}}} \le \frac{\in}{\frac{1}{\sqrt{n}}}]$$

$$= \lim_{n\to\infty} p[\frac{|\hat{\theta}-\theta|}{\frac{1}{\sqrt{n}}} \le \sqrt{n} \in] \qquad \therefore z = \frac{\hat{\theta}-\theta}{\frac{1}{\sqrt{n}}}$$

$$= \lim_{n\to\infty} p[|z| \le \sqrt{n} \in]$$

$$= p[|z| \le \infty]$$

$$= p[-\infty \le z \le \infty]$$

=1

So C-1 is satisfied

Now $\lim_{n\to\infty} v(\hat{\theta})_{\lim_{n\to\infty}} = \lim_{n\to\infty} \frac{1}{n} = 0$

As condition 1 and condition 2 are satisfied so sample mean is consistent estimator for "θ"

Q.2

Show that sample mean is consistent estimator of the parameter μ

If x~N(μ, σ^2)

Solution

As x~N(μ, σ^2)

$\bar{x}$~N$\left(\mu, \frac{\sigma^2}{n}\right)$

i.e

E($\bar{x}$)= E(μ)= μ

Var($\bar{x}$)=v($\hat{\mu}$)= $\delta \bar{x} = \frac{\sigma^2}{n}$

We consider

$$\lim_{n\to\infty} P[|\hat{\theta} - \theta| \le \in]$$

Replacing the values

$$= \lim_{n\to\infty} P[|\bar{x} - \mu| \le \in]$$

Dividing by $\frac{\sigma}{\sqrt{n}}$

$$= \lim_{n\to\infty} P\left[\frac{\left[\left|\bar{x}-\mu\right|\right]}{\frac{\sigma}{\sqrt{n}}} \le \frac{\in}{\frac{\sigma}{\sqrt{n}}}\right]$$

$$= \lim_{n\to\infty} P\left[\frac{\left[\left|\bar{x}-\mu\right|\right]}{\frac{\sigma}{\sqrt{n}}} \le \frac{\sqrt{n}\in}{\sigma}\right]$$

$$= \lim_{n\to\infty} P\left[|Z| \le \frac{\sqrt{n}\in}{\sigma}\right]$$

$$= \lim_{n\to\infty} P\left[|Z| \le \infty\right]$$

$$= \lim_{n\to\infty} P\left[-\infty \le Z \le +\infty\right]$$

=1

So C-1 is satisfied

Now

Now $\lim\limits_{n\to\infty} v(\hat{\theta})_{\lim\limits_{n\to\infty}} = \lim\limits_{n\to\infty} \frac{\sigma^2}{n} = 0$

$= \frac{\sigma^2}{\infty}$

=0

As c-1 and C-2 are satisfied so sample mean is consistent estimator for μ .

Q. 3

Show that from the random sampling of Cauchy distribution

f(x)= $\frac{1}{\pi(1+(x-\theta)^2)}$ $-\infty \le x \le \infty$

Then sample mean is not consistent estimator for the parameter and sample median is a consistent estimator for the parameter.

Solution

(For population mean)

As f(x) $= \frac{1}{\pi(1+(x-\theta)^2)}$ $-\infty \le x \le \infty$

Put z= x - θ

dz= dx

f(z)= $\frac{1}{(1+z^2)}$ $-\infty \le x \le \infty$

We consider

$$\lim_{n\to\infty} p\left[\left|\hat{\theta}-\theta\right| \le \in\right]$$

$$= \lim_{n\to\infty} p\left[\left|\bar{x} - \theta\right| \le \in\right]$$

$$= \lim_{n\to\infty} p\left[\left|z\right| \le \in\right]$$

$$= \lim_{n\to\infty} p\left[- \in \le z \le \in\right]$$

$$= \int_{-\in}^{\in} f(z)dz$$

$$= \frac{1}{\pi} \int_{-\in}^{\in} \frac{1}{1+z^2} dz$$

$$= \frac{1}{\pi} \tan^{-1}(z) \int_{-\in}^{\in}$$

$$= \frac{1}{\pi} \left[\tan^{-1}(\in) - \tan^{-1}(- \in)\right] = \frac{1}{\pi} \left[\tan^{-1}(\in) + \tan^{-1}(\in)\right]$$

$$\frac{2\tan^{-1}(\in)}{\pi} \ne 1$$

As C-1 is not satisfied hence sample mean of Cauchy distribution is not consistant estimator for θ.

Now we consider the median of Cauchy distribution

E($\tilde{x}$)= θ

Var($\tilde{x}$)= $\frac{\pi^2}{4n}$

Now we consider

$$\lim_{n\to\infty} p\left[\left|\hat{\theta} - \theta\right| \le \in\right] = \lim_{n\to\infty} p\left[\left|\tilde{x} - \theta\right| \le \in\right]$$

$$= \lim_{n\to\infty} p\left[\left|\frac{\tilde{x} - \theta}{\pi / 2\sqrt{n}}\right| \le \frac{\in}{\pi / 2\sqrt{n}}\right]$$

$$= \lim_{n\to\infty} p\left[\left|z\right| \le \frac{2\sqrt{n} \in}{\pi}\right]$$

$$= \lim_{n\to\infty} p\left[\left|z\right| \le \infty\right]$$

$$= \lim_{n\to\infty} p\left[-\infty \le z \le \infty\right]$$

=1

So condition 1 is satisfied

Now

$$\lim_{n\to\infty} v(\tilde{x}) = \lim_{n\to\infty} \left(\frac{\pi^2}{4n}\right)$$

$$=\frac{\pi^2}{4(\infty)}$$

$$=\frac{\pi^2}{\infty}$$

=0

As C-1 and C-2 are satisfied so sample median of Cauchy distribution is consistant estimator for parameter θ.

Q. 4

If $X_1,X_2,X_3,\ldots,X_n$ be a random sample of size n from a normal distribution (μ,σ^2)

Then show that:

1) $\bar{x}$ is a consistent estimator of μ
2) $s^2=\frac{\sum(x-\bar{x})^2}{n-1}$ consistant estimator of σ^2

Solution

As

As $x \sim N\left(\mu,\sigma^2\right)$

$$\bar{x} \sim N\left(\mu,\frac{\sigma^2}{n}\right)$$

i.e

$E(\bar{x})= E(\mu)=\mu$

$$Var(\bar{x})=v(\hat{\mu})=\delta^2_{\bar{x}}=\frac{\sigma^2}{n}$$

We consider

$$\lim_{n\to\infty} P[|\hat{\theta}-\theta|\le\in]$$

Replacing the values

$$=\lim_{n\to\infty} P[|\bar{x}-\mu|\le\in]$$

Dividing by $\frac{\sigma}{\sqrt{n}}$

$$=\lim_{n\to\infty} P\left[\frac{[|\bar{x}-\mu|]}{\frac{\sigma}{\sqrt{n}}}\le\frac{\in}{\frac{\sigma}{\sqrt{n}}}\right]$$

$$=\lim_{n\to\infty} P\left[\frac{[|\bar{x}-\mu|]}{\frac{\sigma}{\sqrt{n}}}\le\frac{\sqrt{n}\in}{\sigma}\right]$$

$$=\lim_{n\to\infty} P\left[|Z|\le\frac{\sqrt{n}\in}{\sigma}\right]$$

$= \underset{n\to\infty}{Lim} P[|Z| \le \infty]$

$= \underset{n\to\infty}{Lim} P[-\infty \le Z \le +\infty]$

=1

So C-1 is satisfied

Now

Now $\lim_{n\to\infty} v(\hat{\theta})_{\lim_{n\to\infty}} = \lim_{n\to\infty} \frac{\sigma^2}{n} = 0$

$= \frac{\sigma^2}{\infty}$

=0

As C-1 and C-2 are satisfied so $\bar{x}$ is consistant estimator for μ .

ii) it is given that

$$s^2 = \frac{\sum (x-\bar{x})^2}{n-1}$$

As we know that

E($s^2) = \delta^2$

$$s^2 = \frac{\delta^2}{\delta^2} \quad \frac{\sum (x-\bar{x})^2}{n-1}$$

$$s^2 = \frac{\delta^2}{n-1} \quad \frac{\sum (x-\bar{x})^2}{\delta^2}$$

$$s^2 = \frac{\delta^2}{n-1} \chi^2_{(n-1)}$$

Applying variance on both sides :

$$\text{Var}(s^2) = \frac{\delta^4}{(n-1)^2} \text{ var } (\chi^2_{(n-1)})$$

$$= \frac{\delta^4}{(n-1)^2} 2(\text{n-1}) = \frac{2\delta^4}{(n-1)}$$

Now we consider

$$\lim_{n\to\infty} P[|\theta \hat{\_} \theta| \le \in]$$

$$= \lim_{n\to\infty} p[|s^2 - \delta^2| \le \in]$$

$$= \lim_{n\to\infty} p\left[\frac{|s^2-\delta^2|}{\sqrt{\frac{2\delta^4}{n-1}}} \le \frac{\in}{\left[\sqrt{\frac{2\delta^4}{n-1}}\right]}\right]$$

$$= \lim_{n\to\infty} P\left[\left|\frac{|s^2-\delta^2|}{\sqrt{\frac{2\delta^4}{n-1}}}\right| \le \frac{\sqrt{n-1}\in}{\sqrt{2}\delta^2}\right]$$

$$= \lim_{n\to\infty} p\left[|z| \le \frac{\sqrt{n-1}\in}{\sqrt{2}\delta^2}\right]$$

$$= \lim_{n\to\infty} p\big[|z| \le \infty\big]$$

$$\lim_{n\to\infty} p\big[-\infty \le z \le \infty\big]=1$$

So condition 1 is satisfied

Now

$$\lim_{n\to\infty} v(s^2) = \lim_{n\to\infty} \left(\frac{2\delta^4}{n-1}\right)$$

$$= \frac{2\delta^4}{\infty} = 0$$

Hence C-1 and C-2 are satisfied so $s^2 = \frac{\sum(x-\bar{x})^2}{n-1}$ is an consistent estimator of δ^2.

Q.5

Show that for a sample of size n from a normal distribution (θ, δ^2).Then show that the statistic $\hat{\theta} = \frac{\sum xi}{n+1}$ is an consistant estimator of population parameter θ.

Solution

As we know that

$$\hat{\theta} = \frac{\sum xi}{n+1}$$

Applying expectation on both sides :

$$E(\hat{\theta}) = E\left(\frac{\sum xi}{n+1}\right)$$

$$E(\hat{\theta}) = E\left(\frac{x_1 + x_2 + x_3 .....x_n}{n+1}\right)$$

$$E(\hat{\theta}) = \frac{E(x_1) + E(x_2) + E(x_3) + ... + E(x_n)}{n+1}$$

$$E(\hat{\theta}) = \frac{\theta + \theta + \theta ....\theta}{n+1}$$

$$E(\hat{\theta}) = \frac{n\theta}{n+1}$$

Now applying Var on $\hat{\theta}$

$$\text{Var}(\hat{\theta}) = \frac{v\sum x}{(n+1)^2}$$

$$V(\hat{\theta}) = \frac{V(x1 + x2 + x3.....xn)}{(n+1)^2}$$

$$V(\hat{\theta}) = \frac{V(x1) + V(x2) + V(x3).....V(xn)}{(n+1)^2} = \frac{\delta^2 + \delta^2 + \delta^2.....\delta^2}{(n+1)^2}$$

$$V(\hat{\theta}) = \frac{n\delta^2}{(n+1)^2}$$

Now we consider

$$\lim_{n\to\infty} P[|\theta \hat{\_} \theta| \leq \in]$$

Replacing the values

$$= \lim_{n\to\infty} p\left[\left|\frac{\sum x}{n+1} - \frac{n\theta}{n+1}\right|\right] \leq \in$$

Dividing both sides S.E($\hat{\theta}$)= $\frac{\sqrt{n}\delta}{n+1}$

$$= \lim_{n\to\infty} p\left[\frac{\left|\frac{\sum x}{n+1} - \frac{n\theta}{n+1}\right|}{\frac{\sqrt{n}\delta}{n+1}} \leq \frac{\in}{\frac{\sqrt{n}\delta}{n+1}}\right]$$

$$= \lim_{n\to\infty} P[|z| \leq \frac{(n+1)\in}{\sqrt{n}\delta}]$$

$$= P[|z| \leq \infty]$$

$$= P[-\infty \leq z \leq \infty]$$

=1

So condition 1 is satisfied

Now

$$\lim_{n\to\infty} v(\hat{\theta}) = \lim_{n\to\infty} \left(\frac{n\delta^2}{(n+1)^2}\right) = \frac{\delta^2}{\frac{\infty}{\infty}}$$

=0

Hence C-1 and C-2 are satisfied so $\hat{\theta} = \frac{\sum xi}{n+1}$ is a consistent estimator of population parameter θ